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Representation of an MV-algebra by its triangular norms

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Abstract

Any given complete and atomic Boolean algebra can be embedded into a dense MV-algebra in such a way that the Boolean algebra formed by the idempotent elements of the MV-algebra is isomorphic to the given Boolean algebra. This extended MV-algebra can thereafter be represented by a collection of its own triangular norms.

1. Introduction

A mapping from the nonempty set S into the real closed interval $[0, 1]$ is called a *fuzzy subset* of S [1]. The collection $F(S)$ of all fuzzy subsets of S is a completely distributive lattice under the partial ordering defined as: $\lambda \subseteq \mu$ if and only if $\lambda(x) \leqslant \mu(x)$ for all $x \in S$, where $\lambda, \mu \in F(S)$. One can verify that in this lattice the meet $\cap$, the join $\cup$, the supremum $\cup_i \lambda_i$ and the infimum $\cap_i \lambda_i$ are given by the following formulas:
1. $(\lambda \cap \mu)(x) = \min\{\lambda(x), \mu(x)\}$,
2. $(\lambda \cup \mu)(x) = \max\{\lambda(x), \mu(x)\}$,
3. $(\cap_i \lambda_i)(x) = \inf\{\lambda_i(x)\}_i$,
4. $(\cup_i \lambda_i)(x) = \sup\{\lambda_i(x)\}_i$.

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The subset $C(S)$ of $F(S)$ containing the characteristic functions of S forms a complete and atomic Boolean algebra, which is isomorphic to the Boolean algebra $P(S)$ of all subsets of S. If $0 < \alpha \leqslant 1$, and $x \in S$, then the fuzzy subset $(x)_\alpha$ of S that maps x to α and the remaining elements of S onto 0 is a typical atom of $C(S)$. Summarizing these results, we remark that every complete and atomic Boolean algebra can always be embedded into a completely distributive dense lattice.

In order to generalize the triangle inequality of a metric space, triangular norms were first proposed by Menger ([2]). Initially, these were binary compositions on the real interval $[0, 1]$. Schweizer and Sklar revived these unattended compositions and studied them extensively in [3,4]. After a prolonged experimentation, the final definition of a triangular norm in its general form has now stabilized as follows.

Definition 1.1. Let $\langle L, \leqslant \vee, \wedge\rangle$ be a lattice with 0 and 1. A binary composition T on L is a triangular norm (or, simply a t-*norm*) on L if the following four axioms are satisfied:

(T1) $(xTy)Tz = xT(yTz)$, (associative)
(T2) $xTy = yTx$, (commutative)
(T3) $y \leqslant z \Rightarrow xTy \leqslant xTz$, (isotone)
(T4) $xT1 = x$, (1 is the identity)

for all x, y, z in L.

The meet $\wedge$ is a t-norm on L. The composition N on L defined by

$$xNy = \begin{cases} x \wedge y & \text{if } x \vee y = 1, \\ 0, & \text{otherwise} \end{cases}$$

is a t-norm on L. If $\langle A, +, \cdot, \sim\rangle$ is an MV-algebra, then the product $x \cdot y$ is a t-norm on A such that $x \cdot y \leqslant x \wedge y$. This inequality indicates a general feature, as can be seen below.

Suppose that T and S are two triangular norms on L. If $xTy \leqslant xSy$ for all x, $y \in L$, then we write $T \leqslant S$. Under this ordering, the set τ of all t-norms on L forms an extremely rich partially ordered set. One can easily verify that if T is any triangular norm on L, then $N \leqslant T \leqslant \wedge$. Thus, the meet on L is the greatest t-norm on L. In fact, $T = \wedge$ if and only if every element of L is an idempotent under T.

We point out that in Definition 1.1 no role is played by 0. However we have $xT0 \leqslant x \wedge 0 = 0$ and so $xT0 = 0$ for any $x \in L$.

2. Special triangular norms on a dense lattice

In this section we single out a particular class of triangular norms on a given dense lattice.

Theorem 2.1. *Let* $\langle L, \vee, \wedge, \leqslant \rangle$ *be a lattice with* 0 *and* 1, *and let* $c \in L$. *Then the binary composition* T_c *on L defined by*

$$xT_c y = \begin{cases} 0 & \text{if } x \neq 1 \neq y \text{ and } x \wedge y \leqslant c, \\ x \wedge y, & \text{otherwise} \end{cases}$$

is a triangular norm on L.

Proof. We point out in the beginning that $T_c \leqslant \wedge$. For the sake of convenience, we write T for T_c. Clearly, T satisfies (T2) and (T4).

(T3): Let $x, y, z \in L$ and $y \leqslant z$. We show that $xTy \leqslant xTz$. If $x = 1$ or $y = 1$, the result is obvious. So, let $x \neq 1 \neq y$. If $xTy = 0$, then there remains nothing to prove. So, let $xTy \neq 0$. Then, $xTy = x \wedge y \neq 0$. If $x \wedge z \leqslant c$, then $x \wedge y \leqslant x \wedge z \leqslant c$. We would then get the contradiction $xTy = 0$. Therefore, $x \wedge z \not\leqslant c$. We then get $xTz = x \wedge z \geqslant x \wedge y = xTy$.

(T1): Let $x, y, z \in L$. We show that $(xTy)Tz = xT(yTz)$. If $1 \in \{x, y, z\}$, then the result is obvious. So, let $1 \notin \{x, y, z\}$. We break the problem into four cases.

Case 1: Let $xTy = 0 = yTz$. Then, $(xTy)Tz = 0 = xT(yTz)$.

Case 2: Let $xTy = 0 < yTz$. We get $xT(yTz) = xT(y \wedge z) \leqslant xTy = 0 = (xTy)Tz$.

Case 3: Let $yTz = 0 < xTy$. This is the same as Case 2.

Case 4: Let $xTy > 0 < yTz$. Then, $(xTy)Tz = (x \wedge y)Tz$ and $xT(yTz) = xT(y \wedge z)$. If $x \wedge y \wedge z \leqslant c$, then $(x \wedge y)Tz = 0 = xT(y \wedge z)$ and we are done. So, let $x \wedge y \wedge z \not\leqslant c$. Then we get $(x \wedge y)Tz = (x \wedge y) \wedge z$ and $xT(y \wedge z) = x \wedge (y \wedge z)$. This completes the proof of (T1) as well as the theorem.

Theorem 2.2. *Let* $\langle L, \vee, \wedge, \leqslant \rangle$ *be a dense lattice with* 0 *and* 1 *and let* $c, d \in L$. *Then*

$$c \leqslant d \iff T_d \leqslant T_c.$$

Proof. Firstly, let $c \leqslant d$. Let $x, y \in L$. If $x = 1$, then $xT_d y = y = xT_c y$. If $y = 1$, then $xT_d y = x = xT_c y$. If $x \neq 1 \neq y$ and $x \wedge y \leqslant d$, then $xT_d y = 0 \leqslant xT_c y$. If $x \neq 1 \neq y$ and $x \wedge y \not\leqslant d$, then $xT_d y = x \wedge y = xT_c y$. Conversely, let $T_d \leqslant T_c$. We show that $c \leqslant d$. If either $c = 0$ or $d = 1$, there remains nothing to prove. So, let $0 < c$ and $d < 1$. If possible, suppose that $c \not\leqslant d$. We deal with two cases.

Case 1: Let $c \neq 1$. We have $cT_d c \leqslant cT_c c$, which gives the contradiction $c \leqslant 0$. Consequently Case 1 is impossible.

Case 2: Let $c = 1$. Because L is dense, we can pick up $t \in L$ such that $d < t < 1$. Now, we have $tT_d t \leqslant tT_1 t$. This, again, gives the contradiction $t \leqslant 0$. The only way to come out of this deadlock is to accept that $c \leqslant d$.

3. Embedding a Boolean algebra into a dense MV-algebra

Theorem 3.1. *Any complete and atomic Boolean algebra is the subalgebra of all the idempotent elements of some dense* MV-*algebra.*

Proof. Let M be the set of all atoms of the given complete and atomic Boolean algebra B. The set $F(M)$ of all fuzzy subsets of M is already a completely distributive lattice under meet $\cup$ and join $\cup$, as mentioned in Section 2. For $\lambda, \mu \in F(M)$, define the members $\lambda + \mu$, $\lambda \cdot \mu$ and $\tilde{\lambda}$ of $F(M)$ as follows for any $x \in M$:$(\lambda + \mu)(x) = \min(1, \lambda(x) + \mu(x))$,$(\lambda \cdot \mu)(x) = \max(0, \lambda(x) + \mu(x) - 1)$,-$\tilde{\lambda}(x) = 1 - \lambda(x)$.

Clearly, $(\lambda \cdot \tilde{\mu}) + \mu = \lambda \cup \mu$, and $(\lambda + \tilde{\mu}) \cdot \mu = \lambda \cap \mu$. One verifies that the structure $\langle F(M), +, \cdot, \sim\rangle$ forms a dense MV-algebra in which 1 and $\emptyset$ are unity and zero respectively, where $1(x) = 1$ and $\emptyset(x) = 0$ for all $x \in M$. We get $\xi_A + \xi_A = \xi_A$, where ξ_A is the characteristic function of $A \subseteq M$. On the other hand, if $\lambda + \lambda = \lambda$, where $\lambda \in F(M)$, then A turns out to be a characteristic function of some subset of M. As a consequence, the set $C(M)$ of all characteristic functions of subsets of M is exactly the Boolean subalgebra of the idempotent elements of $F(M)$.

Now, the Boolean algebra $P(M)$ of all subsets of M under the set inclusion is isomorphic to $C(M)$ under the identification map:

$$A \to \xi_A, \quad A \subseteq M.$$

However, $P(M)$ is isomorphic to B, because B is complete and atomic. Hence, one can identify B with $C(M)$.

4. Representation of an MV-algebra by its triangular norms

Lastly, we show how the MV-algebra $F(M)$, which we constructed in the proof of Theorem 1 of Section 3, can be represented by a collection of its own triangular norms.

Theorem 4.1. *If $F(M)$ is the* MV-*algebra of all fuzzy subsets of the set M of all atoms of the complete and atomic Boolean algebra B, then the partially ordered set*

$$\tau_M = \{T_\lambda : \lambda \in F(M)\}$$

of triangular norms of the type T_λ on $F(M)$ as defined in Theorem 2.1, is an MV-*algebra under the compositions:*

$$\begin{aligned} T_\lambda + T_\mu &= T_{\lambda \cdot \mu}, \\ T_\lambda \cdot T_\mu &= T_{\lambda + \mu}, \\ \tilde{T}_\lambda &= T_{\tilde{\lambda}}. \end{aligned}$$

Furthermore, the MV-*algebras $F(M)$ and τ_M are isomorphic.*

Proof. Define $T_\lambda \vee T_\mu = (T_\lambda \cdot \tilde{T}_\mu) + T_\mu$, and $T_\lambda \wedge T_\mu = (T_\lambda + \tilde{T}_\mu) \cdot T_\mu$. Then we get $T_\lambda \vee T_\mu = T_{(\lambda + \tilde{\mu}) \cdot \mu} = T_{\lambda \cap \mu}$, and $T_\lambda \wedge T_\mu = T_{(\lambda \cdot \tilde{\mu}) + \mu} = T_{\lambda \cup \mu}$.

It is now easy to verify that τ_M is an MV-algebra with zero T_1, unity $T_\emptyset$, join operation $\vee$, and meet operation $\wedge$. Because, by Theorem 2.2, $T_\lambda \leqslant T_\mu \iff \mu \subseteq \lambda \iff \lambda \cap \mu = \mu \iff T_{\lambda\cap\mu} = T_\mu \iff T_\lambda \vee T_\mu = T_\mu$, we see that the partial ordering $\leqslant$ on τ_M, as defined in Section 1, tallies with the intrinsic partial ordering on the MV-algebra τ_M.

Lastly, consider the mapping

$$f: F(M) \rightarrow \tau_M \quad \text{defined by} \quad f(\lambda) = \tilde{T}_\lambda, \ \lambda \in F(M).$$

We have

$$f(\lambda + \mu) = \tilde{T}_{(\lambda+\mu)} = T_{\tilde{\lambda}\cdot\tilde{\mu}} = T_{\tilde{\lambda}} + T_{\tilde{\mu}} = \tilde{T}_\lambda + \tilde{T}_\mu = f(\lambda) + f(\mu),$$

$$f_{(\emptyset)} = T_{\tilde{\emptyset}} = T_1,$$

$$f_1 = T_{\tilde{1}} = T_\emptyset,$$

and so, f is a homomorphism of $F(M)$ onto τ_M. Further, by Theorem 2.2, we see that $f(\lambda) = f(\mu) \Rightarrow T_{\tilde{\lambda}} = T_{\tilde{\mu}} \Rightarrow \tilde{\lambda} = \tilde{\mu} \Rightarrow \lambda = \mu$, and so, f is an isomorphism too.

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